Motion of four-dimensional rigid body around a fixed point: an elementary approach. I.

A.M. Perelomov *

Departamento de Física Teórica, Universidad de Zaragoza, E-50009 Zaragoza, Spain

Abstract

The goal of this note is to give the explicit solution of Euler-Frahm equations for the Manakov four-dimensional case by elementary means. For this, we use some results from the original papers by Schottky [Sch 1891], Kötter [Koe 1892], Weber [We 1878], and Caspary [Ca 1893]. We hope that such approach will be useful for the solution of the problem of n-dimensional top.

1. The equations of motion for a rigid body in a four-dimensional Euclidean space with a fixed point coinciding with the center of mass (and also for the n-dimensional case) are the generalization of famous Euler's equations. They were found first by Frahm [Fr 1874][1] and they have the form

$$\dot{l}_{ij} = \sum_{k=1}^{4} (l_{ik}\,\omega_{kj} - \omega_{ik}\,l_{kj}), \quad \omega_{ij} = c_{ij}\,l_{ij}, \quad l_{ij} = -l_{ji}, \quad i,j = 1,\ldots,4. \quad (1)$$

Here $c_{ij} = I_{ij}^{-1}$, the dot denotes the derivative with respect to time t, and l_{ik}, ω_{jk}, and I_{ik} are components of angular momentum, angular velocity and principal momenta of inertia tensors, respectively.

*On leave of absence from Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia. Current e-mail address: perelomo@dftuz.unizar.es

[1] The problem of generalization of Euler's equations was posed by Cayley [Ca 1846].

In this paper we consider completely integrable Manakov's case [Ma 1977], when quantities c_{ij} have the form[2]

$$c_{ij} = \frac{b_i - b_j}{a_i - a_j}. \tag{2}$$

In a number of papers (see [AM 1982], [Ha 1983], [AM 1988], and references therein) so called method of linearization on the Jacobian of a spectral curve defined by the characteristic polynomial of one of the matrix in the Lax pair was used. However, as it was mention in [AM 1988], "this approach has remained unsatisfactory; indeed (i) finding such families of Lax pairs often requires just as much ingenuity and luck as to actually solve the problem; (ii) it often conceals the actual geometry of the problem".

So, in the present note we return to the original Schottky–Kötter approach [Sch 1891], [Koe 1892]. In our opinion, this elementary and natural approach is more adequate for the problem under consideration. We hope that it will be useful also for the more complicated problem of n-dimensional top at $n > 4$.

Let us remind that in the paper [Sch 1891] the problem under consideration was reduced to the Clebsch problem [Cl 1871] of the motion of a rigid body in an ideal fluid[3]. For the special cases, the last problem was integrated explicitly by Weber [We 1878] and by Kötter [Koe 1892].

However, the Clebsch problem is related not to $so(4)$ Lie algebra but to the $e(3)$ Lie algebra – the Lie algebra of motion of the three-dimensional Euclidean space. Hence, it is important to extend the Schottky–Kötter approach to give the solution in $so(4)$ covariant form. Here we give such a solution using the elementary means[4].

2. Note first at all that equations (1) are Hamiltonian with respect to the Poisson structure for the $so(4)$ Lie algebra – the Lie algebra of rotations of the four-dimensional Euclidean space,

$$\{l_{ij}, l_{km}\} = l_{im}\,\delta_{jk} - l_{ik}\,\delta_{jm} + l_{jk}\,\delta_{im} - l_{jm}\,\delta_{ik}. \tag{3}$$

[2] Note that for the "physical" rigid body $c_{ij} = I_{ij}^{-1}$, $I_{ij} = I_i + I_j$. In this paper we consider a general integrable case when quantities c_{ij} and I_{ij} are arbitrary.

[3] This result was rediscovered one century later in the paper [Bo 1986].

[4] A special $so(4)$ case with tensor l_{jk} of rank 2 was integrated explicitly by Moser [Mo 1980].

The Hamiltonian is given by the formula

$$H = \frac{1}{2} \sum_{j<k}^{4} c_{jk} \, l_{jk}^2 \, , \tag{4}$$

where quantities c_{ij} are given by formula (2), and equations (1) may be written in the form

$$\dot{l}_{jk} = \{H, l_{jk}\}. \tag{5}$$

Let us remind that equations (1) have four integrals of motion

$$H_0 = l_{12} \, l_{34} + l_{23} \, l_{14} + l_{31} \, l_{24} = h_0, \tag{6}$$

$$H_1 = \sum_{j<k}^{4} l_{jk}^2 = h_1, \;\; H_2 = \sum_{j<k}^{4} (a_j + a_k) l_{jk}^2 = h_2, \;\; H_3 = \sum_{j<k}^{4} a_j a_k l_{jk}^2 = h_3. \tag{7}$$

Note that H_0 and H_1 are the Casimir functions of $so(4)$-Poisson structure, and the manifold $\mathcal{M}_h$ defined by equations (6) – (7) is an affine part of two-dimensional Abelian manifold (see Appendix by Mumford to the paper [AM 1982])[5]. Then formula (5) defines Hamiltonian vector field on $\mathcal{M}_h$.

The main result of this note is the following one: by elementary means, it is shown that the dynamical variables $l_{jk}(t)$ are expressed in terms of Abelian functions $f_{j4}(u_1, u_2)$, $f_{kl}(u_1, u_2)$, $f_0(u_1, u_2)$, and $g(u_1, u_2)$ related to genus two algebraic curve

$$y^2 = \prod_{j=0}^{4} (x - d_j), \qquad d_0 = 0, \qquad d_4 = d_1 d_2 d_3, \tag{8}$$

with arguments depending linearly on time.

Theorem. *Solution of equations* (1) *has the form*

$$\begin{aligned} m_j &= l_{kl} = g(u_1, u_2) \, (\alpha_j \, f_{kl}(u_1, u_2) + \beta_j \, f_{j4}(u_1, u_2)), & (9) \\ n_j &= l_{j4} = g(u_1, u_2) \, (\gamma_j \, f_{kl}(u_1, u_2) + \delta_j \, f_{j4}(u_1, u_2)). & (10) \end{aligned}$$

Here (j, k, l) *is a cyclic permutation of* (1,2,3), α_j, β_j, γ_j, δ_j, *and* d_j *are algebraic functions of integrals of motion and quantities* a_j *and* b_k. *Explicit expressions for them are given by* (24)–(26), (34), (35), (41), *and* (44).

[5] I am grateful to A. N. Tyurin for the explanation of algebraic geometry related to this Appendix.

Proof. The key problem is the "uniformization" of the manifold $\mathcal{M}_h$, i.e., finding of the "good" coordinates on it. The proof consists of several steps.

A. Following Kötter [Koe 1892] and using the linear change of variables m_j and n_j to new variables ξ_j and η_j, we transform equation (7) to the more appropriate form:

$$\sum_{j=1}^{3}\left(\xi_j^2+\eta_j^2\right)=0, \qquad \sum_{j=1}^{3}\xi_j\eta_j=0, \qquad \sum_{j=1}^{3}\left(d_j\xi_j^2+d_j^{-1}\eta_j^2\right)=0. \tag{11}$$

For this, following Schottky [Sch 1891], let us introduce the three-dimensional vector $\mathbf{l}(s)$ depending on parameter s:

$$\mathbf{l}(s)=(l_1(s),l_2(s),l_3(s)), \qquad l_j(s)=\sqrt{s_{j4}}\,m_j+\sqrt{s_{kl}}\,n_j, \tag{12}$$

where

$$m_j=l_{kl}, \qquad n_j=l_{j4}, \qquad s_{jk}=(s-a_j)(s-a_k), \tag{13}$$

and $\{j,k,l\}$ is a cyclic permutation of $\{1,2,3\}$. It is easy to check that the function

$$f(s)=\mathbf{l}(s)^2=\sum_{j=1}^{3}l_j(s)\,l_j(s) \tag{14}$$

does not depend on time. So, it is the generating function of integrals of motion

$$f(s)=h_1s^2-h_2s+h_3+2\,h_0\sqrt{G(s)}, \qquad G(s)=\prod_{j=1}^{4}(s-a_j). \tag{15}$$

From formulae (12) and (14) it is easy to get the Lax representation [6]

$$\dot{L}(s)=[L(s),M(s)], \tag{16}$$

where $L(s)$ and $M(s)$ are antisymmetric matrices of the third order corresponding to vectors $\mathbf{l}(s)$ and $\mathbf{m}(s)$,

$$\mathbf{m}(s)=(m_1(s),m_2(s),m_3(s)), \qquad m_j(s)=\sqrt{s_{kl}}\,m_j+\sqrt{s_{j4}}\,n_j, \tag{17}$$

[6] However, this representation does not need for the proof of Theorem. For the generalization of such representation for the n-dimensional case see [Fe 2000].

$$L(s) = \begin{pmatrix} 0 & l_3 & -l_2 \\ -l_3 & 0 & l_1 \\ l_2 & -l_1 & 0 \end{pmatrix}, \qquad M(s) = \begin{pmatrix} 0 & m_3 & -m_2 \\ -m_3 & 0 & m_1 \\ m_2 & -m_1 & 0 \end{pmatrix}. \tag{18}$$

The equation $f(s) = 0$ is equivalent to the algebraic equation of fourth degree $F(s) = \prod_{j=1}^{4}(s - s_j) = 0$, where

$$F(s) = \left[\left(h_1 s^2 - h_2 s + h_3\right)^2 - 4\, h_0^2\, G(s)\right] / \left(h_1^2 - 4h_0^2\right). \tag{19}$$

This equation has four roots s_1, s_2, s_3, and s_4 that, in general, are complex ones. To them correspond four complex vectors

$$\mathbf{l}^{(p)} = \mathbf{l}(s_p)/\sqrt{F'(s_p)}, \qquad p = 1, 2, 3, 4, \tag{20}$$

(here $F'(s)$ is the derivative of $F(s)$) but only two of them, for example $\mathbf{l}^{(1)}$ and $\mathbf{l}^{(2)}$, are linearly independent, and

$$\left(\mathbf{l}^{(p)}\right)^2 = \sum_{k=1}^{3}\left(l_k^{(p)}\right)^2 = 0,\ \ p = 1, 2, 3, 4;\ \sum_{p=1}^{4}\left(l_k^{(p)}\right)^2 = 0,\ \ k = 1, 2, 3. \tag{21}$$

Let us introduce also the vectors ξ and η by the formulae [7]

$$\xi_j = l_j^{(1)} + i l_j^{(2)}, \qquad \eta_j = l_j^{(1)} - i l_j^{(2)}. \tag{22}$$

Using (12) and (22) we may express m_j and n_j in terms of ξ_j and η_j

$$m_j = \alpha_j \xi_j + \beta_j \eta_j, \qquad n_j = \gamma_j \xi_j + \delta_j \eta_j, \tag{23}$$

where

$$\alpha_j = \frac{\sqrt{s_{kl}^{(2)}/F'(s_2)} - i\sqrt{s_{kl}^{(1)}/F'(s_1)}}{\Delta_j^{(3)}}, \quad \beta_j = \frac{\sqrt{s_{kl}^{(2)}/F'(s_2)} + i\sqrt{s_{kl}^{(1)}/F'(s_1)}}{\Delta_j^{(3)}},$$
$$\gamma_j = \frac{\sqrt{s_{j4}^{(2)}/F'(s_2)} - i\sqrt{s_{j4}^{(1)}/F'(s_1)}}{\Delta_j^{(3)}}, \quad \delta_j = \frac{\sqrt{s_{j4}^{(2)}/F'(s_2)} + i\sqrt{s_{j4}^{(1)}/F'(s_1)}}{\Delta_j^{(3)}}, \tag{24}$$

[7] As it was noted by Yu. N.Fedorov, there is relation of these vectors to the problem of geodesics on two-dimensional ellipsoid with half-axes $\sqrt{d}_j$, j=1,2,3. Namely, ξ may be considered as a tangent vector to geodesics and $i\eta$ as a normal vector to this geodesics.

$$\Delta_j^{(p)} = \frac{\sqrt{s_{j4}^{(q)}\, s_{kl}^{(r)} - s_{j4}^{(r)}\, s_{kl}^{(q)}}}{\sqrt{F'(s_q)\, F'(s_r)}}. \tag{25}$$

Here (j,k,l) and (p,q,r) are cyclic permutations of (1, 2, 3).

Now it is easy to check that equations (7) take the form of three Kötter's quadrics (11), where

$$\sqrt{d_j} = \frac{\Delta_j^{(1)} - i\Delta_j^{(2)}}{\Delta_j^{(3)}}, \qquad \frac{1}{\sqrt{d_j}} = -\frac{\Delta_j^{(1)} + i\Delta_j^{(2)}}{\Delta_j^{(3)}}. \tag{26}$$

B. Following [Koe 1892], let us show that the manifold defined by equations (11) may be "uniformized" by means of the Weierstrass Wurzelfunctionen related to the hyper-elliptic curve (8) that are defined as

$$P_j(z_1, z_2) = \sqrt{(z_1 - d_j)(z_2 - d_j)}, \qquad j,k = 0,1,2,3,4, \tag{27}$$

$$P_{jk}(z_1, z_2) = \frac{P_j P_k}{(z_1 - z_2)} \left(\frac{\sqrt{R(z_1)}}{(z_1 - d_j)(z_1 - d_k)} - \frac{\sqrt{R(z_2)}}{(z_2 - d_j)(z_2 - d_k)} \right). \tag{28}$$

These sixteen functions $P_j(z_1, z_2)$ and $P_{jk}(z_1, z_2)$ satisfy a lot of identities. All of them may be obtained from definitions (27) and (28) (for details, see [We 1878], [Koe 1892], and [Ca 1893]). [8] Here we give only few of them which are useful for us:

$$\sum_{j=1}^{3} c_j \left(\frac{P_{kl}^2}{(s - d_k)(s - d_l)} + \frac{P_{j4}^2}{(s - d_j)(s - d_4)} \right) = \frac{s}{\prod_{j=1}^{4}(s - d_j)}, \tag{29}$$

$$\sum_{j=1}^{3} \tilde{c}_j\, P_{j4}^2 = d_4, \qquad \sum_{j=1}^{3} d_j \tilde{c}_j\, P_{kl}^2 = P_0^2, \tag{30}$$

$$\sum_{j=1}^{3} c_j\, P_{j4} P_{kl} = 0, \qquad \sum_{j=1}^{3} \tilde{c}_j\, P_{j4} P_{kl} = -P_0, \tag{31}$$

$$\sum_{j=1}^{3} c_j \left(d_j^{-1} P_{j4}^2 + d_j P_{kl}^2 \right) = 0, \tag{32}$$

[8] See also modern survey [BEL 1997].

where

$$\tilde{c}_j = \frac{1}{(d_j - d_k)(d_j - d_l)}, \qquad c_j = \frac{d_j - d_4}{(d_j - d_k)(d_j - d_l)}. \tag{33}$$

It is known (see [We 1878]) that $P_j(z_1, z_2)$ and $P_{jk}(z_1, z_2)$ up to the factors are the ratio of the theta functions with half-integer theta characteristics [9]

$$P_j(z_1, z_2) = f_j(u_1, u_2) = \frac{\theta_j(u_1, u_2)}{\theta(u_1, u_2)}, \quad P_{kl}(z_1, z_2) = f_{kl}(u_1, u_2) = \frac{\theta_{kl}(u_1, u_2)}{\theta(u_1, u_2)}, \tag{34}$$

$$\begin{array}{rcl}
\theta_{23}(u_1, u_2) & = & \theta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2), \qquad \theta_{31}(u_1, u_2) = \theta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2), \\
\theta_{12}(u_1, u_2) & = & \theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u_1, u_2), \qquad \theta_{14}(u_1, u_2) = \theta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u_1, u_2), \\
\theta_{24}(u_1, u_2) & = & \theta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u_1, u_2), \qquad \theta_{34}(u_1, u_2) = \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u_1, u_2), \\
\theta_0(u_1, u_2) & = & \theta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u_1, u_2), \qquad \theta(u_1, u_2) = \theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u_1, u_2).
\end{array} \tag{35}$$

Here

$$\theta(u_1, u_2) = \sum_{n_1, n_2} \exp\{i\pi(n_1(2u_1 + n_1\tau_{11} + n_2\tau_{12}) + n_2(2u_2 + n_1\tau_{21} + n_2\tau_{22}))\}, \tag{36}$$

and τ_{jk} are elements of period matrix.

The comparison (11) with (29)–(33) shows that

$$\xi_j = \sqrt{c_j} g\, P_{kl}, \qquad \eta_j = \sqrt{c_j} g\, P_{j4}, \tag{37}$$

where g is an unknown function.

C. The rest part of the proof is the uniformization of equation (6).

Let us substitute expressions (23) for m_j and n_j into equation (6). Then by using of (24) and (25) we transform it to the form

$$H_0 = \sum_{j=1}^{3} \left(A_j \left(\xi_j^2 - \eta_j^2 \right) + B_j \xi_j \eta_j \right) = h_0, \tag{38}$$

[9] We give here just one series of such expressions. Relative other series, see [Koe 1892].

where

$$A_j = \alpha + \beta d_j + \gamma {d_j}^{-1}, \qquad B_j = \delta \left(d_j + d_j^{-1} \right). \tag{39}$$

Here α, β, γ, and δ are algebraic functions of h_0, h_1, h_2, h_3, a_j, and b_j.

This sum may be calculated by using of (24), (29)–(33). The result is

$$H_0 = \frac{(1 - \varepsilon P_0)^2}{4\varepsilon d_4}\, g^2 = h_0, \tag{40}$$

where

$$\varepsilon = \frac{\sqrt{d_4}\left(\sqrt{(s_3 - s_1)(s_2 - s_4)} - \sqrt{(s_2 - s_3)(s_1 - s_4)}\right)}{\sqrt{(s_1 - s_2)(s_3 - s_4)}}. \tag{41}$$

From this we obtain

$$g = (1 - \varepsilon f_0)^{-1}, \qquad \xi_j = g\sqrt{c_j}\, f_{kl}, \qquad \eta_j = g\sqrt{c_j}\, f_{j4}. \tag{42}$$

The fact of linear dependence of arguments u_1 and u_2 on time t follows as from the algebraic geometrical approach [AM 1982] as from the old Kötter approach [Koe 1892].

This completes the proof of Theorem.

Acknowledgements. The main part of this paper was completed during my visit to Max-Planck Institute for Mathematics, Bonn. I am grateful to the stuff of this Institute and also to the Department of Theoretical Physics of Zaragoza University for hospitality.

References

[AM 1982] Adler M. and van Moerbeke P., *The algebraic integrability of geodesic flow on* $SO(4)$, Inv. Math. **67**, 297–326; with appendix by D. Mumford, pp. 326 – 331

[AM 1988] Adler M. and van Moerbeke P., *The Kowalewski and Hénon-Heiles motions as Manakov geodesic flows on* $SO(4)$ *– a two-dimensional family of Lax pairs*, CMP **113**, 659–700

[BEL 1997] Bukhstaber V.M., Enolskii V.Z. and Leikin D.V. *Hyperelliptic Kleinian functions and applications*, Solitons, geometry, and topology: on crossroad, 1-33, Amer. Math. Soc. Transl. Ser 2, 179, Amer. Math. Soc., Providence, RI

[Bo 1986] Bobenko A.I., *Euler equations on the Lie algebras $e(3)$ and $so(4)$. Isomorphism of integrable cases*, Funct. Anal. Appl. **20**, 53–56

[Ca 1893] Caspary F., *Sur une nouvelle manière d'établir les relations algébriques qui ont lieu entre les fonctions hyperelliptiques de première espèce*, Ann. Sci. Ec. Norm. Sup., 3 Ser., **10**, 253–294

[Ca 1846] Cayley A., *Sur quelques propriétés des détérminants gauches*, J. Reine Angew. Math. **32**, 119-123

[Cl 1871] Clebsch A., *Über die Bewegung eines Körpers in einer Flüssigkeit*, Math. Ann. **3**, 238-261

[Fe 2000] Fedorov Yu.N., *Integrable systems, Poisson pencils, and hyperelliptic Lax pairs*, Reg. Chaot. Dyn. **5**, No.2, 171–180

[Fr 1874] Frahm F., *Über gewisse Differentialgleichungen*, Math. Ann. **8**, 35–44

[Ha 1983] Haine L., *Geodesic flow on $SO(4)$ and abelian surfaces*, Math. Ann. **263**, No.4, 435–472

[Koe 1892] Kötter F., *Über die Bewegung eines festen Körpers in einer Flüssigkeit*, I, II, J. Reine Angew. Math. **109**, 51–81, 89–111

[Ma 1977] Manakov S.V., *Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body*, Funct. Anal. Appl. **10**, 328–329

[Mo 1980] Moser Ju., *Geometry of quadrics and spectral theory*, in: *Chern Symposium 1979*, 147-188

[Sch 1891] Schottky F., *Über das analytische Problem der Rotation eines starren Körpers im Raume von vier Dimensionen*, Sitzungber. König. Preuss. Akad. Wiss. zu Berlin, 227–232

[We 1878] Weber H., *Anwendung der Thetafunctionen zweiter Veränderlicher auf die Theorie der Bewegung eines festen Körpers in einer Flüssigkeit*, Math. Ann. **14**, 173–206